# Perturbation-Theoretical Integral Representation and the High-Energy Behavior of the Scattering Amplitude*

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The high-energy behavior of the scattering amplitude is investigated in detail in the ladder approximation by means of the perturbation-theoretical integral representation. It is shown that this representation without subtraction is very useful and can describe the Regge behavior. The main results of the "multiperipheral model" are reproduced in this approach, and its connection with the Regge pole is clarified in an elegant way. It is also pointed out that we can by no means exclude the possibility of non-Regge behavior in the ladder approximation, at least within the present technique.

## I. INTRODUCTION

IT is an open question whether or not the scattering amplitude exhibits the Regge behavior in the high-energy limit. Since it is extremely difficult to discuss the exact amplitude without introducing hypotheses, it will be very important and suggestive to investigate the simplest approximation, i.e., the ladder approximation, in detail. It seems to be generally believed that the scattering amplitude exhibits the Regge behavior in the ladder approximation at least for spinless particles, but we shall see that this is not conclusive. Lee and Sawyer[1] have shown that the partial-wave amplitude is meromorphic in $\mathrm{Re}l > -\frac{3}{2}$, where $l$ stands for the complex angular momentum. Their result is doubtful, however, because as previously shown,[2] a mathematically unjustifiable manipulation (term-by-term simultaneous rotation of energy variables) is made in their proof. Polkinghorne,[3] and Federbush and Grisaru[4] have shown that the sum of the leading terms gives the Regge-type high-energy behavior, but of course such a heuristic method does not provide any proof for the high-energy behavior of the scattering amplitude itself.

On the other hand, an interesting approach has been developed by Bertocchi, Fubini, and Tonin,[5] who call the ladder approximation the "multiperipheral model." They have started from an integral equation for the off-the-mass-shell absorptive part[6] and shown that it indeed exhibits the Regge behavior in the asymptotic region. The Regge trajectory is determined by a homogeneous equation, which is equivalent to the Bethe-Salpeter equation that is decomposed into partial waves.[5,7] We must notice, however, that the Regge behavior has been introduced as an "ansatz," and they have verified it only in a self-consistent way. Thus non-Regge behaviors are not excluded also in their analysis.

Now, the purpose of the present paper is to investigate the properties of the Feynman amplitude for the scattering of two scalar particles in the ladder approximation. We shall make use of the perturbation-theoretical integral representation[8] because this approach will be the most standard method. We have seen that the perturbation-theoretical integral representation is very useful for discussing the Bethe-Salpeter equation in the ladder approximation.[2,9]

In Sec. II the integral equation for the Feynman amplitude is converted into an integral equation for the weight function. In Sec. III we consider the asymptotic equation of the latter, which has the Regge-type solution. The equation then reduces exactly to the same equation as that for the weight function in the Bethe-Salpeter equation as is expected. The connection between the high-energy behavior and the analyticity in the $l$ plane is clarified in an elegant way. In Sec. IV we point out that the asymptotic equation admits also non-Regge-type solutions. At the present stage we cannot exclude the possibility of these solutions at all. In Sec. V we consider the iterative solution of our inhomogeneous integral equation. We find that it is meaningful *except for* a certain region which determines the high-energy behavior of the Feynman amplitude. A special discussion is made for the case in which the exchanged-meson mass is zero. The final section is devoted to discussions on our results.

## II. INTEGRAL REPRESENTATION

We consider the Feynman amplitude for the elastic scattering of two scalar particles having masses $m_1$ and $m_2$. They exchange a scalar meson having mass $\mu$. Let $2k$, $q$, and $p$ be the total momentum, the relative mo-

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[1] B. W. Lee and R. F. Sawyer, Phys. Rev. **127**, 2266 (1962).

[2] N. Nakanishi, Phys. Rev. **130**, 1230 (1963).

[3] J. C. Polkinghorne, J. Math. Phys. **4**, 503 (1963).

[4] P. G. Federbush and M. T. Grisaru, Ann. Phys. (N. Y.) **22**, 263 (1963).

[5] L. Bertocchi, S. Fubini, and M. Tonin, Nuovo Cimento **25**, 626 (1962). See also C. Ceolin, F. Duimio, R. Stroffolini, and S. Fubini, *ibid.* **26**, 247 (1962); D. Amati, A. Stanghellini, and S. Fubini, *ibid.* **26**, 896 (1962).

[6] One should note that their absorptive part is not equal to the usual one-off-the-mass shell because of the appearance of anomalous thresholds.

[7] B. W. Lee and A. R. Swift, Nuovo Cimento **27**, 1272 (1963).

[8] N. Nakanishi, Progr. Theoret. Phys. (Kyoto) **26**, 337 (1961); *ibid.* Errata **28**, 406 (1962). See also N. Nakanishi, Institute for Advanced Study, Princeton, 1963 (unpublished lecture note).

[9] M. Ida and K. Maki, Progr. Theoret. Phys. (Kyoto) **26**, 470 (1961); I. Sato, J. Math. Phys. **4**, 24 (1963).

mentum in the initial state, and that in the final state, respectively. The initial momenta $k+q$ and $k-q$ lie on the mass shells. Then the integral equation for the Feynman amplitude in the ladder approximation is

$$(m_1^2-v)(m_2^2-w)f(v,w,t)$$
$$=\frac{1}{\mu^2-t-i\epsilon}+\frac{\lambda}{\pi^2 i}\int d^4p'\frac{f(v',w',t')}{\mu^2-(p-p')^2-i\epsilon}. \quad (2.1)$$

Here $f(v,w,t)$ stands for the Feynman amplitude apart from a trivial factor, and $\lambda=g^2/(4\pi)^2$, where $g$ is the coupling constant. The invariant squares are as follows:

$$\begin{aligned}(k+q)^2&=m_1^2, & (k+p)^2&=v, & (k+p')^2&=v',\\ (k-q)^2&=m_2^2, & (k-p)^2&=w, & (k-p')^2&=w',\\ (2k)^2&=s, & (p-q)^2&=t, & (p'-q)^2&=t'.\end{aligned} \quad (2.2)$$

We shall confine ourselves to considering the unphysical region

$$s<(m_1+m_2)^2, \quad (2.3)$$

because we are mainly interested in the high-energy behavior in the $t$ channel.

Now, we introduce the perturbation-theoretical integral representation:

$$f(v,w,t)=\int_0^1 dy\int_{-1}^1 dz\int_{0-}^\infty d\gamma\frac{\varphi(y,z,\gamma)}{\{\gamma-(1-y)[\frac{1}{2}(1+z)(v-m_1^2)+\frac{1}{2}(1-z)(w-m_2^2)]-y(t-\mu^2)-i\epsilon\}^3}. \quad (2.4)$$

This representation is valid for each term of the perturbation expansion, and so within the convergence radius of the series.[10] We shall, however, use it also for the case in which the perturbational series is no longer convergent. This extension is supported by the fact that all solutions of the Bethe-Salpeter equation, which is of course nonperturbational, have the perturbation-theoretical integral representation of two variables.[2] The existence of a solution which cannot be represented as (2.4) is extremely unlikely because the solution of (2.1) is expected to be unique.

We substitute (2.4) in (2.1) and carry out the momentum integration by the standard method. We then have[11]

$$\begin{aligned}F(v,w,t)&\equiv(\pi^2 i)^{-1}\int d^4p'[\mu^2-(p-p')^2-i\epsilon]^{-1}\\&\qquad\times\{\gamma'-(1-y')[\tfrac{1}{2}(1+z')(v'-m_1^2)+\tfrac{1}{2}(1-z')(w'-m_2^2)]-y'(t'-\mu^2)-i\epsilon\}^{-3}\\&=\frac{1}{2}\int_0^1(1-x)^2dx(Q-i\epsilon)^{-2},\end{aligned} \quad (2.5)$$

where

$$\begin{aligned}Q\equiv{}&(1-x)\gamma'+[x+(1-x)^2y']\mu^2+(1-x)^2(1-y')^2\rho(z')\\&-x(1-x)\{(1-y')[\tfrac{1}{2}(1+z')(v-m_1^2)+\tfrac{1}{2}(1-z')(w-m_2^2)]+y'(t-\mu^2)\},\end{aligned} \quad (2.6)$$

with

$$\rho(z)\equiv\tfrac{1}{2}(1+z)m_1^2+\tfrac{1}{2}(1-z)m_2^2-\tfrac{1}{4}(1-z^2)s>0. \quad (2.7)$$

Next, we combine $(m_1^2-v-i\epsilon)^{-1}(m_2^2-w-i\epsilon)^{-1}$ with $F$ by means of the Feynman identity:

$$(m_1^2-v-i\epsilon)^{-1}(m_2^2-w-i\epsilon)^{-1}F(v,w,t)$$
$$=\int_0^1 dy\int_{-1}^1 dz\int_{0-}^\infty d\gamma\frac{K(y,z,\gamma;y',z',\gamma')}{\{\gamma-(1-y)[\frac{1}{2}(1+z)(v-m_1^2)+\frac{1}{2}(1-z)(w-m_2^2)]-y(t-\mu^2)-i\epsilon\}^3}, \quad (2.8)$$

with

$$\begin{aligned}K(y,z,\gamma;y',z',\gamma')\equiv{}&\frac{1}{2}\int_0^1(1-x)^2dx\int_{-1}^1 d\xi\int_0^1 d\zeta\,\zeta(1-\zeta)[\zeta x(1-x)+(1-\zeta)]^{-4}\delta\left(z-\frac{\zeta x(1-x)(1-y')z'+(1-\zeta)\xi}{\zeta x(1-x)(1-y')+(1-\zeta)}\right)\\&\times\delta\left(y-\frac{\zeta x(1-x)y'}{\zeta x(1-x)+(1-\zeta)}\right)\delta'\left(\gamma-\frac{\zeta\{(1-x)\gamma'+[x+(1-x)^2y']\mu^2+(1-x)^2(1-y')^2\rho(z')\}}{\zeta x(1-x)+(1-\zeta)}\right).\end{aligned} \quad (2.9)$$

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[10] A lower bound $\lambda_0$ of the convergence radius can be easily calculated according to the method of N. Nakanishi, J. Math. Phys. (to be published). For example, in the case of $m_1\geq m_2\geq\mu$, $s\leq 0$, $t\leq 0$, $v\leq m_1^2$, and $w\leq m_1^2$, we have

$$\lambda_0=[\Gamma(1/3)]^{-3}\mu^4(m_1^2+\mu^2)^{-1}.$$

[11] In order to avoid the appearance of a complex momentum, we first put $s\leq(m_1-m_2)^2$, and after the momentum integration is carried out $s$ should be analytically continued to the region $(m_1-m_2)^2<s<(m_1+m_2)^2$.

Carrying out the integrations over $\zeta$ and $\xi$, we obtain

$$K(y,z,\gamma;y',z',\gamma')$$
$$=\tfrac{1}{2}y(1-y)\theta(y'-y)\theta\left(R(z,z')-\frac{y(1-y')}{y'(1-y)}\right)$$
$$\times\int_0^1(1-x)^2dx\,\delta'(x(1-x)y'\gamma-y\{(1-x)\gamma'$$
$$+[x+(1-x)^2y']\mu^2+(1-x)^2(1-y')^2\rho(z')\}),\quad(2.10)$$

where

$$R(z,z')\equiv(1\mp z)/(1\mp z')\quad\text{for}\quad z\gtrless z'.\quad(2.11)$$

The second $\theta$ function in (2.10) implies $y'\geq y$, hence we may omit $\theta(y'-y)$. Thus, according to the uniqueness theorem of the perturbation-theoretical integral representation,[12] we obtain the integral equation for the weight function:

$$\varphi(y,z,\gamma)=(1-y)\delta(\gamma)+\lambda\int_0^1dy'\int_{-1}^1dz'\int_{0-}^\infty d\gamma'$$
$$\times K(y,z,\gamma;y',z',\gamma')\varphi(y',z',\gamma')\quad(2.12)$$

with (2.10), (2.11), and (2.7).

### III. REGGE BEHAVIOR

We consider the high-energy behavior of $f(v,w,t)$ for the $t$ channel. According to (2.4), when $|t|$ is very large, the dominant contribution should come from a neighborhood of $y=0$. Hence we suppose that $y$ and $y'$ are very small in (2.12). Since we expect that $\varphi(y,z,\gamma)$ is singular at $y=0$, the inhomogeneous term may be neglected. We then have the "asymptotic equation":

$$\varphi(y,z,\gamma)\simeq\lambda\int_0^1dy'\int_{-1}^1dz'\int_{0-}^\infty d\gamma'$$
$$\times K_0(y,z,\gamma;y',z',\gamma')\varphi(y',z',\gamma'),\quad(3.1)$$

where

$$K_0(y,z,\gamma;y',z',\gamma')\equiv\tfrac{1}{2}y\theta(y'R(z,z')-y)$$
$$\times\int_0^1x^{-2}dx\,\delta'(y'\gamma-yg(\gamma',z',x))\quad(3.2)$$

and

$$g(\gamma',z',x)\equiv x^{-1}[\gamma'+(1-x)\rho(z')]+(1-x)^{-1}\mu^2.\quad(3.3)$$

The symbol $\simeq$ in (3.1) means that the equality is required only for the dominant singularity at $y=0$ but not for less singular parts.

Because of the very simple dependence of $K_0$ on $y$ and $y'$, one can easily see that (3.1) has a solution expressed as

$$\varphi(y,z,\gamma)\simeq y^{-l-1}\varphi(z,\gamma).\quad(3.4)$$

[12] N. Nakanishi, Phys. Rev. **127**, 1380 (1962).

Here we assume Re$l>-1$, otherwise we cannot neglect the inhomogeneous term. Since if we use (3.4) the integral (2.4) becomes undefined for Re$l\geq0$, we employ the following distribution defined by Schwartz[13]:

$$Y_m(y)\equiv[\Gamma(m)]^{-1}\,\mathrm{Pf}(y^{m-1})_{y>0},\ \text{for } m\neq0,-1,\cdots$$
$$\equiv\delta^{(l)}(y),\ \text{for } m=-l=0,-1,-2,\cdots,\quad(3.5)$$

where Pf denotes Hadamard's finite part, namely, it indicates to discard the divergent part of the integral in a mathematically consistent manner. $\int dyY_m(y)h(y)$ is an entire function of $m$, where $h(y)$ is an arbitrary test function.

Now, instead of (3.4), we put

$$\varphi(y,z,\gamma)\simeq Y_{-l}(y)\varphi_l(z,\gamma).\quad(3.6)$$

Then $\varphi_l(z,\gamma)$ should satisfy a homogeneous equation

$$\varphi_l(z,\gamma)=\lambda\int_{-1}^1dz'\int_{0-}^\infty d\gamma'K_l(z,\gamma;z',\gamma')\varphi_l(z',\gamma'),\quad(3.7)$$

with

$$K_l(z,\gamma;z',\gamma')\equiv\frac{1}{2}\int_0^1x^{-2}dx[g(\gamma',z',x)]^{-l-2}$$
$$\times\frac{\partial}{\partial\gamma}\{\gamma^{l+1}[\theta(\gamma)-\theta(\gamma-R(z,z')g(\gamma',z',x))]\}.\quad(3.8)$$

It is remarkable that (3.7) is exactly the same as the integral equation for the weight function of the Bethe-Salpeter equation with the angular momentum $l$, and with $n=l+1$.[2] According to the result obtained in that case, we have[14]

$$\lim_{\gamma\to\infty}\varphi_l(z,\gamma)=0.\quad(3.9)$$

The high-energy behavior of $f(v,w,t)$ for the $t$ channel can be easily evaluated by inserting (3.6) into (2.4). Since

$$\int_0^1dy\frac{y^{m-1}}{(A-yt)^3}\simeq\int_0^\infty dy\frac{y^{m-1}}{(A-yt)^3}$$
$$=\tfrac{1}{2}\Gamma(m)\Gamma(-m+3)(-t)^{-m}A^{m-3},\quad(3.10)$$

for $3>\mathrm{Re}m>0$, we obtain

$$f(v,w,t)\simeq\tfrac{1}{2}\Gamma(l+3)(-t)^l\int_{-1}^1dz\int_{0-}^\infty d\gamma$$
$$\times\frac{\varphi_l(z,\gamma)}{[\gamma-\frac{1}{2}(1+z)(v-m_1^2)-\frac{1}{2}(1-z)(w-m_2^2)-i\epsilon]^{l+3}}\quad(3.11)$$

[13] L. Schwartz, *Theorie des Distributions* (Herman and Cie., Paris, 1950), Chapter II.

[14] The proof was explicitly given only in the case in which $l$ is an integer, but its generalization is straightforward if we assume that $\varphi_l(z,\gamma)$ does not increase faster than $\gamma^{l+1-\epsilon}$, $(\epsilon>0)$.

by means of analytic continuation with respect to $m$. Thus $f(v,w,t)$ exhibits the Regge behavior because $l$ is determined as a function of $s$ by the eigenvalue problem (3.7). This result exactly corresponds to that of Bertocchi, Fubini, and Tonin.[5]

It is interesting to see how this high-energy behavior is related to the analyticity in the $l$ plane without recourse to the Regge analysis. According to Khuri,[15] the analyticity of the partial-wave amplitude for $\mathrm{Re}\,l>-\frac{1}{2}$ is essentially the same as that of the coefficient of a power series expansion with respect to a linear function of $t$. We expand (2.4) as

$$f(v,w,t)=\sum_{l=0}^{\infty}(t-\mu^2)^l f_l(v,w)\,, \tag{3.12}$$

with

$$f_l(v,w)\equiv\tfrac{1}{2}(l+1)(l+2)\int_{0-}^{1}dy\int_{-1}^{1}dz\int_{0-}^{\infty}d\gamma\frac{y^l\varphi(y,z,\gamma)}{\{\gamma-(1-y)[\frac{1}{2}(1+z)(v-m_1^2)+\frac{1}{2}(1-z)(w-m_2^2)]-i\epsilon\}^{l+3}}. \tag{3.13}$$

We regard $f_l(v,w)$ as an analytic function of $l$. Putting $\gamma=(1-y)\alpha$ and

$$\psi_l(z,\alpha)\equiv\int_0^1 dy\cdot y^l(1-y)^{-l-2}\varphi(y,z,\,(1-y)\alpha)\,, \tag{3.14}$$

we have

$$f_l(v,w)=\tfrac{1}{2}(l+1)(l+2)\int_{-1}^{1}dz\int_{0-}^{\infty}d\alpha \times\frac{\psi_l(z,\alpha)}{[\alpha-\frac{1}{2}(1+z)(v-m_1^2)-\frac{1}{2}(1-z)(w-m_2^2)-i\epsilon]^{l+3}}. \tag{3.15}$$

Thus the analyticity of $f_l(v,w)$ with respect to $l$ is determined by that of $\psi_l(z,\alpha)$. Now, suppose that $\varphi(y,z,\gamma)$ behaves like

$$\varphi(y,z,\gamma)\simeq Y_{-l'}(y)\,\varphi_{l'}(z,\gamma) \tag{3.16}$$

in a neighborhood of $y=0$ as in (3.6). Then (3.14) leads to

$$\psi_l(z,\alpha)\simeq\int_0^1 dy\cdot y^l[\Gamma(-l')]^{-1}\,\mathrm{Pf}y^{-l'-1}\varphi_{l'}(z,\alpha) =[\Gamma(-l')]^{-1}(l-l')^{-1}\varphi_{l'}(z,\alpha)\,. \tag{3.17}$$

Thus $\psi_l(z,\alpha)$ has a simple pole at $l=l'$. This is a Regge pole, and its position is determined by the Bethe-Salpeter Eq. (3.7). Thus the connection between the high-energy behavior and the Regge pole is understood in an elegant way in terms of the perturbation-theoretical integral representation.

### IV. POSSIBLE NON-REGGE BEHAVIOR

The reasoning of the previous section is based on the presumption (3.4) or (3.6). In general, we should write

$$\varphi(y,z,\gamma)\simeq F(y)\,\varphi(z,\gamma)\,, \tag{4.1}$$

where $F(y)$ should be self-reproducing, apart from less singular parts. Since

$$y\int_{y/R}^{1}\delta'(y'\gamma-yg)F(y')dy' =y\frac{\partial}{\partial\gamma}\int_{y/R}^{1}\delta(y'\gamma-yg)\frac{F(y')}{y'}dy' \simeq g^{-1}\frac{\partial}{\partial\gamma}\left\{F\left(\frac{yg}{\gamma}\right)[\theta(\gamma)-\theta(\gamma-Rg)]\right\}. \tag{4.2}$$

$F(y)$ must satisfy

$$F(y)\simeq aF(hy)+byF'(hy)\,, \tag{4.3}$$

where $a$, $b$, and $h$ ($>0$) are independent of $y$. The general solution of (4.3) will be given by

$$F(y)\simeq cy^{m-1}(\ln 1/y)^n(\ln\ln 1/y)^r\cdots, \tag{4.4}$$

where $c, m, n, r, \cdots$ are constants. If we put

$$n=r=\cdots=0\,, \tag{4.5}$$

$F(y)$ reduces to the solution given in the previous section, but there is no reason why (4.5) gives the correct solution. As is seen from (4.2), *the homogeneous equation for* $\varphi(z,\gamma)$ *depends only on* $m$ *but not on* $n$, $r$, *etc.*

For simplicity, we consider the case[16]

$$F(y)=cy^{m-1}(\ln 1/y)^n\,,\quad (n\neq 0)\,. \tag{4.6}$$

Then the behavior of $f_l$ in the $l$ plane is determined by

$$\int_0^1 dy\cdot y^l y^{m-1}(\ln 1/y)^n=\frac{\Gamma(n+1)}{(l+m)^{n+1}}. \tag{4.7}$$

Thus $f_l$ has a multiple pole or a branch point at $l=-m$. The corresponding high-energy behavior will be proportional to

$$(-t)^{-m}[\ln(-t)]^n\,, \tag{4.8}$$

where $m$ is a function of $s$.

[15] N. N. Khuri, Phys. Rev. Letters **10**, 420 (1963); Phys. Rev. **132**, 914 (1963).

[16] We have suppressed the symbol Pf for simplicity.

## V. PARTIAL SOLUTION

We could not decide whether or not $f(v,w,t)$ exhibits the Regge behavior from the self-consistency of (2.12) at a neighborhood of $y=0$. Of course, if we knew the exact solution of (2.12), such a problem would no longer occur. Hence, it is desirable to investigate the properties of the solution of (2.12) in detail.

The integration over $x$ in (2.10) can be easily carried out, and the result is

$$K(y,z,\gamma;y',z',\gamma')=\tfrac{1}{2}y^{-1}(1-y)\theta\left(R(z,z')-\frac{y(1-y')}{y'(1-y)}\right)\frac{\partial I}{\partial\beta},\quad(5.1)$$

with

$$I\equiv\frac{\beta-\gamma'-\mu^2}{\gamma'+\sigma}\,\frac{\theta(\beta-\gamma'-\mu^2-2\mu(\gamma'+\sigma)^{1/2})}{[(\beta-\gamma'-\mu^2)^2-4\mu^2(\gamma'+\sigma)]^{1/2}},\quad(5.2)$$

$$\beta\equiv(y'/y)\gamma\geq\gamma,\quad(5.3)$$

$$\sigma\equiv y'\mu^2+(1-y')^2\rho(z').\quad(5.4)$$

When $\gamma\gg\max(|s|,m_1^2,m_2^2,\mu^2)$, the kernel tends to

$$K\sim\tfrac{1}{2}(1-y)\theta\left(R(z,z')-\frac{y(1-y')}{y'(1-y)}\right)\frac{\delta(y'\gamma-y\gamma')}{\gamma'+\sigma}.\quad(5.5)$$

Hence we shall have

$$\varphi(y,z,\gamma)=O(\gamma^{-1})\quad\text{as}\quad\gamma\to\infty,\quad(5.6)$$

a result which is consistent with (3.9).

Now, the kernel (5.1) is very singular so that we cannot apply the Fredholm theory as in the Bethe-Salpeter equation. Instead, we have an inequality

$$\gamma\geq y[(\gamma')^{1/2}+\mu]^2,\quad(5.7)$$

on account of (5.2) with (5.3). Let $\varphi^{(n)}(y,z,\gamma)$ be the $n$th-order iteration term. Then (5.7) yields

$$\varphi^{(n)}(y,z,\gamma)=0,\ \text{unless}\ \gamma\geq y\left(\frac{1-y^{n/2}}{1-y^{1/2}}\right)^2\mu^2,\ (n\geq1).\quad(5.8)$$

Thus, when $\mu\neq0$, the exact solution $\varphi(y,z,\gamma)$ can be obtained for $\gamma<y(1-y^{1/2})^{-2}\mu^2$ by a finite order iteration. The behavior at a neighborhood of $y=0$ cannot be obtained.

The case of $\mu=0$ is especially simple. In this case we have

$$\partial I/\partial\beta=y\delta(y'\gamma-y\gamma')[\gamma'+(1-y')^2\rho(z')]^{-1}.\quad(5.9)$$

If we put

$$\varphi(y,z,\gamma)=\delta(\gamma)(1-y)\chi(y,z),\quad(5.10)$$

then (2.12) reduces to

$$\chi(y,z)=1+\tfrac{1}{2}\lambda\int_0^1dy'\int_{-1}^1dz'\times\theta\left(R(z,z')-\frac{y(1-y')}{y'(1-y)}\right)\frac{\chi(y',z')}{y'(1-y')\rho(z')}.\quad(5.11)$$

This equation was also given by Nishijima[17] in different notations. He converted it into a differential equation by employing somewhat complicated transformations. Instead, in (5.11) we put

$$y=(1+\eta)^{-1},\quad y'=(1+\eta')^{-1}.\quad(5.12)$$

Then (5.11) becomes

$$\psi(\eta,z)=1+\tfrac{1}{2}\lambda\int_{-1}^1dz'\int_0^{\eta R(z,z')}d\eta'\frac{\psi(\eta',z')}{\eta'\rho(z')},\quad(5.13$$

where

$$\psi(\eta,z)\equiv\chi(y,z).\quad(5.14)$$

Now, assume that $\psi(\eta,z)$ vanishes at $\eta=0$ in such a way that the $\eta'$ integral in (5.13) converges at $\eta'=0$. Then $\psi(\eta,z)$ is continuous at $z=\pm1$, and moreover the inhomogeneous term gives

$$\psi(\eta,\pm1)=1.\quad(5.15)$$

Therefore $\psi(\eta,z)$ differs from zero independently of $\eta$ at neighborhoods of $z=\pm1$. This is inconsistent with our assumption. We thus see that (5.13) has no solution regular at $\eta=0$.[17] This situation physically corresponds to the fact that the initial state cannot be a plane-wave state in the presence of Coulomb force. We cannot, however, exclude the possibility that (5.13) might have a solution singular at $\eta=0$ in the sense of (3.5), in contrast with the case of $\mu\neq0$ in which such a singular solution is excluded by the iterative solution well defined at $\eta=0$ (i.e., $y=1$).

It is noteworthy that the *homogeneous* equation corresponding to (5.13) can be solved exactly. If we put

$$\psi(\eta,z)=\eta^ng_n(z),\quad(\mathrm{Re}\,n>0),\quad(5.16)$$

then $g_n(z)$ should satisfy

$$g_n(z)=\frac{\lambda}{2n}\int_{-1}^1dz'[R(z,z')]^n\frac{g_n(z')}{\rho(z')}.\quad(5.17)$$

This equation is exactly the same as Cutkosky's equation.[18] Hence it can be converted into a differential equation.

## VI. DISCUSSION

We have investigated the high-energy behavior of the Feynman amplitude in the $t$ channel in terms of the perturbation-theoretical integral representation. The most remarkable result of our analysis will be that the Regge behavior or more general high-energy behavior can be consistently described by the perturbation-theoretical integral representation (2.4) *without subtraction*. The validity of no subtraction is due to the presence of the parameter $y$.

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[17] K. Nishijima, Progr. Theoret. Phys. (Kyoto) **14**, 203 (1955).
[18] R. E. Cutkosky, Phys. Rev. **96**, 1135 (1954).

Consider the Mandelstam representation[19]

$$\int_0^\infty ds' \int_0^\infty dt' \frac{\sigma(s',t')}{(s'-s-i\epsilon)(t'-t-i\epsilon)}, \tag{6.1}$$

where we have suppressed two remaining terms for simplicity. If $\sigma(s',t')$ does not vanish at infinity, it is necessary to make subtractions. Main subtraction terms are single dispersion integrals of $s$ and $t$. On the other hand, the perturbation-theoretical integral representation for the scattering amplitude[8] is

$$\int_0^1 dz \int_0^\infty d\alpha \frac{\varphi(z,\alpha)}{\alpha - zs - (1-z)t - i\epsilon} \tag{6.2}$$

apart from two other terms. As was shown previously,[12] no single dispersion terms appear for (6.2) when subtractions are made. This fact exactly corresponds to the above mentioned situation, namely, the possible singularity of $\varphi(z,\alpha)$ at $z=0$ or at $z=1$ can describe the high-energy behavior concerning a single variable $s$ or $t$. Recently Khuri[15] has shown that one can take out the Regge pole contributions from (6.1) under certain assumptions. If this is the case, we see that no subtraction is needed for (6.2).

Now, our second result is a negative one. We have seen that even in the ladder approximation we cannot prove the Regge behavior at least within the present technique. There may appear multiple poles or branch points in the $l$ plane. Of course, branch points may appear by summing specially selected Feynman graphs,[20] but such a procedure seems to be very artificial. Our result concerns the *simplest* approximation, hence it is very suggestive. In this connection it might be interesting to note that recent experiments[21] have denied the simple Regge pole hypothesis.

Finally, we comment on the possibility of rotating the integration path of the relative energy $p_0$ on the basis of the integral representation (2.4). If such a rotation were possible, (2.1) would become a Fredholm equation and hence we could obtain the exact solution. In the unphysical region $(m_1-m_2)^2<s<(m_1+m_2)^2$, it is evident that either $k$ or $q$ must be a complex vector, so that we have complex singularities in the $p_0$ plane. For $s<(m_1-m_2)^2$, the amplitude can become analytic in the first and third quadrants under certain artificial restrictions.[22] However, the important point is that *the contribution from the infinite quarter-circles may not vanish*[23] if we assume a singularity of $\varphi(y,z,\gamma)$ at $y=0$ just as in Sec. III or IV. Thus we cannot employ the $p_0$ rotation in order to investigate the Regge behavior of the amplitude.

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[22] When $s<0$ and $m_1=m_2\equiv m$, there are no singularities in the first and third quadrants of the $p_0$ plane if $\mu>m$ and $s>-4(\mu^2-m^2)$.

[23] Rather, it may be more appropriate to say that it is not well defined.